Visualizing Monodromy Groups of Torodial Belyĭ Pairs

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Abstract

A Belyĭ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a rational function with at most three critical values; we may assume these values are $\{0,\,1,\,\infty\}$. A Dessin d'Enfant is a planar bipartite graph obtained by considering the preimage of a path between two of these critical values, usually taken to be the line segment from 0 to 1. Such graphs can be drawn on the sphere by composing with stereographic projection: $\beta^{-1}([0,1]) \subseteq \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$. Replacing $\mathbb{P}^1$ with an elliptic curve E, there is a similar definition of a Belyĭ map $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Since $E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$ is a torus, we call (E, β) a toroidal Belyĭ pair. The corresponding Dessin d'Enfant can be drawn on the torus by composing with an elliptic logarithm: $\beta^{-1}([0,1]) \subseteq E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$.

In this project, we are interested in the group $\mathrm{Mon}(\beta) = \mathrm{im}\left[\pi_1(\mathbb{P}^1(\mathbb{C}) - \{0,\,1,\,\infty\}) \to S_N\right]$ called the monodromy group. We layout a quick algorithm to compute these groups by solving a system of ordinary differential equations and present visualizations of their group actions on the sphere.

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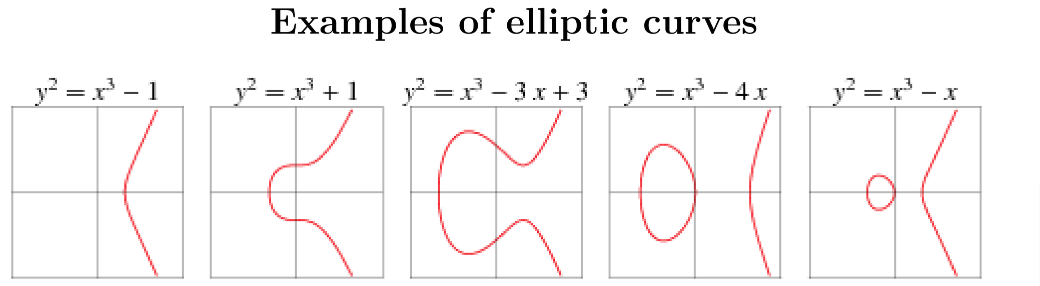
Background

Let X be a compact, connected Riemann surface. There are two examples of interest:

- **The Sphere:** the projective line $\mathbb{P}^1$ may be embedded into the projective plane using the map $\mathbb{P}^1 \to \mathbb{P}^2$ which sends $(x_1 : x_0) \mapsto (x_1 : 0 : x_0)$, so that this curve corresponds to the zeroes of the polynomial $f(x,y) = y$. The set of complex points, namely $X = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$, is a sphere.

- **Elliptic Curves:** an elliptic curve E is a nonsingular projective variety corresponding to the zeroes of the form
$$f(x,y) = \left(y^2 + a_1\,x\,y + a_3\,y\right) - \left(x^3 + a_2\,x^2 + a_4\,x + a_6\right) = 0.$$

Examples of elliptic curves



The surface defined by an Elliptic curve over the complex numbers is equivalent to a torus.

Belyĭ Map: a Belyĭ Map is a rational function $\beta : X \to \mathbb{P}^1(\mathbb{C})$ with at most 3 critical values, which we assume to be $\{0, 1, \infty\}$.

Since X may be viewed as the set of zeroes of a single polynomial $f(x,y)$, we can write $\beta(x,y) = p(x,y)/q(x,y)$ as the ratio of two polynomials $p(x,y)$ and $q(x,y)$.

Some examples include:

$$\beta(x,y) = \frac{y+1}{2} \quad \text{for} \quad E : y^2 = x^3 + 1$$
$$\beta(x,y) = \frac{(y - x^2 - 17x)^3}{2^{14}y} \quad \text{for} \quad E : y^2 + 15xy + 128y = x^3$$
$$\beta(x,y) = \frac{(x-5)y + 16}{32} \quad \text{for} \quad E : y^2 = x^3 + 5x + 10$$

Monodromy Groups

Fix $y_0 \in \mathbb{P}^1(\mathbb{C})$ different from 0, 1, and ∞. For each P_i in the collection of affine points

$$\beta^{-1}(y_0) = \left\{ (x:y:1) \in X \;\middle|\; \begin{array}{r} f(x,y) = 0 \\ p(x,y) - y_0\,q(x,y) = 0 \end{array} \right\} = \{P_1,\, P_2,\, \dots,\, P_N\}$$

there exist unique paths $\widetilde{\gamma}_0^{(i)},\, \widetilde{\gamma}_1^{(i)} : [0,1] \to X$ satisfying

$$\begin{cases} \beta\left(\widetilde{\gamma}_0^{(i)}(t)\right) = y_0\,e^{2\pi\sqrt{-1}t} \\ \widetilde{\gamma}_0^{(i)}(0) = P_i \end{cases} \qquad \begin{cases} \beta\left(\widetilde{\gamma}_1^{(i)}(t)\right) = 1 + (y_0 - 1)\,e^{2\pi\sqrt{-1}t} \\ \widetilde{\gamma}_1^{(i)}(0) = P_i \end{cases}$$

There exist permutations $\sigma_0,\,\sigma_1,\,\sigma_\infty \in S_N$ such that $\widetilde{\gamma}_0^{(i)}(1) = P_{\sigma_0(i)}$, $\widetilde{\gamma}_1^{(i)}(1) = P_{\sigma_1(i)}$, and $\sigma_\infty = \sigma_1^{-1} \circ \sigma_0^{-1}$ for $i = 1,\,2,\,\dots,\,N$. Then $\mathrm{Mon}(\beta) = \langle \sigma_0,\,\sigma_1,\,\sigma_\infty \rangle$ is called the monodromy group of β. It is a transitive subgroup of S_N.

Algorithm

The paths $\widetilde{\gamma}_0^{(i)},\, \widetilde{\gamma}_1^{(i)} : [0,1] \to X$ must also satisfy the system of ordinary differential equations

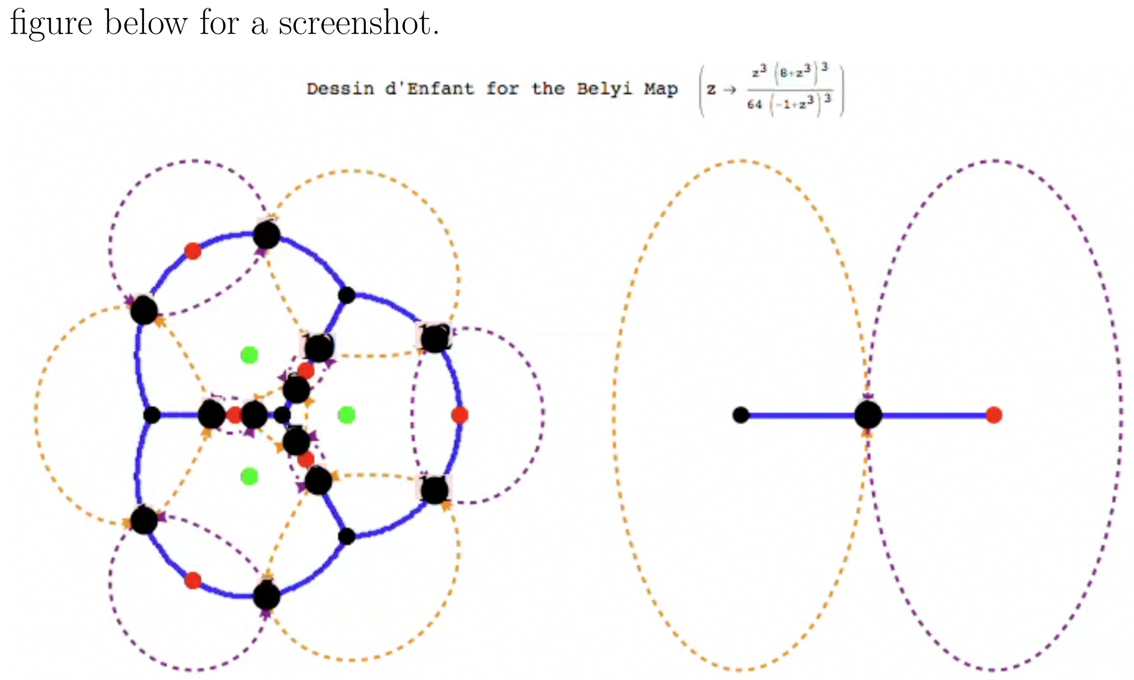
$$\begin{cases} \dfrac{d\widetilde{\gamma}_0^{(i)}}{dt} = \dfrac{2\pi\sqrt{-1}\,p\,q}{q\left(\dfrac{\partial f}{\partial x}\dfrac{\partial p}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial p}{\partial x}\right) - p\left(\dfrac{\partial f}{\partial x}\dfrac{\partial q}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial q}{\partial x}\right)} \begin{bmatrix} -\dfrac{\partial f}{\partial y} \\ +\dfrac{\partial f}{\partial x} \end{bmatrix} \\ \widetilde{\gamma}_0^{(i)}(0) = P_i \end{cases}$$

$$\begin{cases} \dfrac{d\widetilde{\gamma}_1^{(i)}}{dt} = \dfrac{2\pi\sqrt{-1}\,(p-q)\,q}{q\left(\dfrac{\partial f}{\partial x}\dfrac{\partial p}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial p}{\partial x}\right) - p\left(\dfrac{\partial f}{\partial x}\dfrac{\partial q}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial q}{\partial x}\right)} \begin{bmatrix} -\dfrac{\partial f}{\partial y} \\ +\dfrac{\partial f}{\partial x} \end{bmatrix} \\ \widetilde{\gamma}_1^{(i)}(0) = P_i \end{cases}$$

After solving these equations to suitable numerical precision, we choose $\sigma_0,\,\sigma_1,\,\sigma_\infty \in S_N$ as those permutations such that

- $\sigma_0(i) = j$ is that index where the difference $\left|P_j - \widetilde{\gamma}_0^{(i)}(1)\right|$ is least for $i = 1,\,2,\,\dots,\,N$;
- $\sigma_1(i) = j$ is that index where the difference $\left|P_j - \widetilde{\gamma}_1^{(i)}(1)\right|$ is least for $i = 1,\,2,\,\dots,\,N$; and
- $\sigma_\infty = \sigma_1^{-1} \circ \sigma_0^{-1}$.

There is preliminary software which partially does this in `Mathematica`; see figure below for a screenshot.



Animation Featuring Monodromy Action

http://www.math.purdue.edu/~egoins/notes/dessin_explorer.mov

Examples on the Sphere

Say that $X = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$.

- The rational function $\beta(z) = z^N$ is a Belyĭ map of degree N. The monodromy group has the generators
$$\sigma_0 = (1\;2\;\cdots\;N) \qquad \sigma_1 = (1) \qquad \sigma_\infty = (N\;\cdots\;2\;1)$$
Hence the monodromy group is $\mathrm{Mon}(\beta) = \langle \sigma_0,\,\sigma_1,\,\sigma_\infty \rangle = Z_N$.

- The rational function $\beta(z) = (1 - 2\,z)^3\,(1 + 3\,z)^2$ is a Belyĭ map of degree $N = 5$. According to our `Mathematica` code, the monodromy group has the generators
$$\sigma_0 = (1\;2)\,(3\;4\;5) \qquad \sigma_1 = (2\;3) \qquad \sigma_\infty = (1\;2\;5\;4\;3)$$
Hence the monodromy group is $\mathrm{Mon}(\beta) = \langle \sigma_0,\,\sigma_1,\,\sigma_\infty \rangle = S_5$.

- The rational function $\beta(z) = -(z-1)\,(2\,z^2 + 3\,z + 9)^3/729$ is a Belyĭ map of degree $N = 7$. According to our `Mathematica` code, the monodromy group has the generators
$$\sigma_0 = (1\;5\;3)\,(2\;4\;6) \qquad \sigma_1 = (3\;7\;4) \qquad \sigma_\infty = (1\;3\;2\;6\;4\;7\;5)$$
Hence the monodromy group is $\mathrm{Mon}(\beta) = \langle \sigma_0,\,\sigma_1,\,\sigma_\infty \rangle = A_7$.

Examples on the Torus

Say that $X = E(\mathbb{C}) \simeq T^2(\mathbb{R})$.

- Consider $E : y^2 = x^3 + 1$. The rational function $\beta(x,y) = (y+1)/2$ is a Belyĭ map of degree $N = 3$. According to our `Mathematica` code, the monodromy group has the generators
$$\sigma_0 = (1\;2\;3) \qquad \sigma_1 = (1\;2\;3) \qquad \sigma_\infty = (1\;2\;3)$$
Hence the monodromy group is $\mathrm{Mon}(\beta) = \langle \sigma_0,\,\sigma_1,\,\sigma_\infty \rangle = A_3$.

- Consider $E : y^2 = x^3 - x$. The rational function $\beta(x,y) = x^2$ is a Belyĭ map of degree $N = 4$. According to our `Mathematica` code, the monodromy group has the generators
$$\sigma_0 = (1\;3)\;(2\;4) \qquad \sigma_1 = (1\;2\;3\;4) \qquad \sigma_\infty = (1\;2\;3\;4)$$
Hence the monodromy group is $\mathrm{Mon}(\beta) = \langle \sigma_0,\,\sigma_1,\,\sigma_\infty \rangle = Z_4$.

- Consider $E : y^2 = x^3 + x^2 + 16\,x + 180$. The rational function $\beta(x,y) = \left(x^2 + 4\,y + 56\right)/108$ is a Belyĭ map of degree $N = 4$. According to our `Mathematica` code, the monodromy group has the generators
$$\sigma_0 = (1)\;(2\;3\;4) \qquad \sigma_1 = (1\;4\;2\;3) \qquad \sigma_\infty = (1\;2\;3\;4)$$
Hence the monodromy group is $\mathrm{Mon}(\beta) = \langle \sigma_0,\,\sigma_1,\,\sigma_\infty \rangle = S_4$.

Future Work

We have preliminary code in `Mathematica`, which we plan to port to `Sage`. While `Mathematica` solves these systems of differential equations very quickly, it cannot determine the structure of groups very well. On the other hand, `Sage` can determine the structure of groups, but cannot solve systems of differential equations when complex numbers are involved.

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